Comment on "a unified scheme for flavored meson and baryons"

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Abstract

We would comment on the results of the paper "a unified scheme for flavored meson and baryons" (P.C.Vinodkumar , J.N.Panandya, V.M.Bannur, and S.B.Khadkikar Eur. Phys. J. A4(1999)83), and point out some inconsistencies and mistakes in the work for solving the Dirac equation. In terms of an example for a single particle we investigate the reliability of the perturbative method for computing the Coulomb energy and discuss the contribution to the wavefunction at origin from the Coulomb potential. We conclude that the accuracy of their numerical results needs to be reconsidered.

In the recent paper [1], based on a relativistic harmonic confinement model (RHM) with a phenomenological parameterization for the residual color electrostatical potential and using the residual confined one-gluon-exchange potential (COGEP) for the spin-hyperfine interaction, an attempt was made to compose a unified understanding of the spectroscopy of hadrons from light flavors to heavy ones. The authors of Ref.[1] computed the masses of mesons and baryons of different flavor combinations for $q\bar{q}, q\bar{Q}, Q\bar{Q}, qqq, qqQ, qQQ, QQQ$ etc. where q and Q stand for light and heavy flavor quarks respectively, with a unique

confinement strength . In this framework they obtained mass differences of pseudoscalar-to-vector mesons as well as that for 1/2-to-3/2 baryons. They predicted the masses of the S-state flavor mesons and baryons which are not measured yet and evaluated leptonic decay widths of the vector mesons with no further additional parameters. As shown in the tables of [1], their numerical results are in good agreement with the experimental data, much of which are even better than that obtained in terms of some successful models.

The idea is obviously remarkable that one can suppose the effective mechanism for both light and heavy quarks to be the same and all hadron states can be well described in a unified framework. However, unfortunately, we find that there are some evident inconsistencies and mistakes in their work while solving the Dirac equation, using the perturbative method to compute the Coulomb energy and omitting the Coulomb potential to calculate the wavefunction at origin. The ill-treatment greatly influences the reliability of their numerical results.

1. On the solution of the Dirac equation.

The basic Dirac equation (3) in [1] which describes the confined single particle state of the quark under a mean field potential $V(r)$ is the starting point of their calculations. It reads

$$[i\gamma^{\mu}\partial\mu - M_q - V(r)]\psi_q(r) = 0 \tag{1}$$

where V(r) is a potential of form

$$V(r) = \frac{1}{2}(1+\gamma_0)A^2r^2, \tag{2}$$

which includes both scalar and vector parts.

The wavefunction of quark $\psi_q(r)$ was expressed as a bispinor form $[\chi_q, \phi_q]$ which satisfies coupled equations,

$$[E - M_q - A^2r^2]\chi_q = -i\vec{\sigma}\cdot\nabla\phi_q \tag{3}$$

$$[E + M_q]\phi_q = -i\vec{\sigma} \cdot \nabla \chi_q, \tag{4}$$

where M_q is the quark mass parameter and the E is the eigenvalue of energy.

The authors of Ref.[1] introduced an operator U of the form

$$U = \frac{1}{1 + \frac{p^2}{(E+M_q)^2}} \begin{pmatrix} 1 & \frac{\vec{\sigma}\cdot\vec{p}}{(E+M_q)} \\ -\frac{\vec{\sigma}\cdot\vec{p}}{(E+M_q)} & 1 \end{pmatrix}. \tag{5}$$

Then ψ_q can be transformed into a form where the lower component is eliminated

$$U\psi_q = \begin{pmatrix} \chi_q \\ 0 \end{pmatrix}. \tag{6}$$

They gave the following normalization

$$\langle \psi_q | \psi_q \rangle = \langle \chi_q | \chi_q \rangle = 1 \tag{7}$$

and the upper component χ_q satisfies

$$[-\nabla^2 + A^2 r^2 (E + M_q)]\chi_q = (E^2 - M_q^2)\chi_q. \tag{8}$$

However, it is very easy to prove that U is not a unitary operator because

$$U^\dagger U = U U^\dagger = \frac{1}{1 + \frac{p^2}{(E+M)^2}} \neq 1, \tag{9}$$

unless under the extreme non-relativistic limit.

Thus, Eq.(7) is incorrect. Instead, we should have

$$\begin{aligned} \langle \psi_q | \psi_q \rangle &= \langle \psi_q | U^\dagger (1 + \frac{p^2}{(E + M^2)}) U | \psi_q \rangle \\ &= \langle \chi_q | (1 + \frac{p^2}{(E + M^2)}) | \chi_q \rangle = 1. \end{aligned} \tag{10}$$

Eq.(10) gives the normalization condition for χ_q. In fact, we do not need the transformation operator U, because $E + M_q \neq 0$, from Eq. (4) we have

$$\phi_q = \frac{\vec{\sigma} \cdot \vec{p}}{(E + M)} \chi_q. \tag{11}$$

Substituting it into Eq.(3), a corresponding equation similar to Eq.(8) for χ_q can be obtained immediately. The normalization condition Eq.(10) is satisfied naturally. However, an important difference manifests. The lower component of the ψ_q is not zero and cannot be neglected in the calculation of matrix element, which would have a great influence to the numerical results of Ref.[1].

It is not difficult to understand this inconsistency. Instead of U we can take an unitary operator $U_0 = \sqrt{1+\frac{p^2}{(E+M_q)^2}}U$.i.e.

$$U_0 = \frac{1}{\sqrt{1+\frac{p^2}{(E+M_q)^2}}} \begin{pmatrix} 1 & \frac{\vec{\sigma}\cdot\vec{p}}{(E+M_q)} \\ -\frac{\vec{\sigma}\cdot\vec{p}}{(E+M_q)} & 1 \end{pmatrix}. \tag{12}$$

Using U we can eliminate the lower component of ψ_q too, but instead of the form of Eq. (6) we have

$$U_0\psi_q = \sqrt{1+\frac{p^2}{(E+M_q)^2}} \begin{pmatrix} \psi_q^{up} \\ 0 \end{pmatrix}. \tag{13}$$

Evidently, ψ_q^{up} satisfies the following equation:

$$U_0[i\gamma^\mu\partial\mu - M_q - V(r)]U_0^\dagger\psi_q^{up}(r) = 0. \tag{14}$$

Because the commutation relation between the operator p and r is not zero, the equation for $\psi_q^{up}(r)$ is much more complicated than Eq.(8).

Actually, it is not difficult to obtain an analytical solution of the Dirac Eq.(1) directly. According to the general way for solving the Dirac equation with central potential [2], let

$$\psi_{q,jm}(\vec{r}) = \begin{pmatrix} i\frac{G(r)}{r}\Omega_{jlm}(\frac{\vec{r}}{r}) \\ -\frac{F(r)}{r}\Omega_{jl'm}(\frac{\vec{r}}{r}) \end{pmatrix}, \tag{15}$$

where j is the total angular momentum quantum number, l and l' are the orbital angular momentum quantum numbers, Ω_{jlm} is the well-known spherical spinor and

$$l' = 2j-1 = \begin{cases} 2(l+\frac{1}{2})-l = l+1 & for\ j = l+\frac{1}{2} \\ 2(l-\frac{1}{2})-l = l-1 & for\ j = l-\frac{1}{2}. \end{cases} \tag{16}$$

Now we insert the expression (16) into (1) and can obtain the differential equation for the radial parts G and F:

$$\frac{dG(r)}{dr} = -\frac{\kappa}{r}G(r) + [E + M_q]F(r) \tag{17}$$

$$\frac{dF(r)}{dr} = \frac{\kappa}{r}F(r) - [E - M_q - A^2r^2]G(r), \tag{18}$$

where κ is a quantum number used frequently in solving Dirac equation with central potential. It is defined as

$$\kappa = \mp(j + \frac{1}{2}) = \begin{cases} -(l+1) & for\ j = l + \frac{1}{2} \\ l & for\ j = l - \frac{1}{2} \end{cases}. \tag{19}$$

From Eq. (17) we can obtain

$$F = \frac{1}{E + M_q}[\frac{dG}{dr} + \frac{\kappa}{r}G]. \tag{20}$$

We insert the expression for F into Eq.(18) and obtain a differential equation for G of the form

$$-\frac{d^2G}{dr^2} + \frac{\kappa(\kappa+1)}{r^2}G + (E + M_q)A^2r^2G = (E^2 - M_q^2)G. \tag{21}$$

It is easy to see that, if we consider κ as an angular momentum quantum number l, Eq.(21) will be the same as the equation for the reduced radial wave function $u(r) = \frac{\chi_q(r)}{r}$. Therefore, we can obtain the solution for E, G and then F by using the similar method as that used by the authors of Ref.[1]. The proper normalization condition is

$$\int_0^\infty (G^2 + F^2)dr = 1 \tag{22}$$

2. On the negative energy state

The potential under consideration is of the form of three-dimensional harmonic oscillator and we can easily obtain solution for eq. (8). The authors of ref.[1] gave the single

particle energy as (see Eq. (9) in [1])

$$E_N = \pm\sqrt{M_q^2 + (2N+3)\Omega_N(q)}. \tag{23}$$

Then they claimed that "following Dirac, the negative energy state is interpreted as antiparticle". In fact, this statement is incorrect. Because a necessary condition for Eq.(8) having bound-state- solution is $E + M_q > 0$ and by the definition,

$$\Omega_N = A(E_N + M_q)^{\frac{1}{2}}, \tag{24}$$

therefore, in this case Eq.(23) only has an unique positive real solution. Taking the negative sign in Eq. (23), we obtain $E = -M_q$, which is not a solution and moreover its absolute value is not equal to the positive solution either. Thus, the negative energy solution cannot be interpreted as one for an antiparticle.

3. On the computation of the Coulomb energy

In [1], a residual Coulomb potential

$$V_{coul}(q_i q_j) = \frac{\alpha_s^{eff}(\mu)}{r} \tag{25}$$

was introduced and then the Coulomb part of the energy was computed perturbatively using the confinement basis. They gave

$$\epsilon_n(q_i q_j)_{coul} = \langle N|V_{coul}|N\rangle. \tag{26}$$

Below we will examine the reliability of the result by this approximate computational method. For convenience, we are not going to repeat their calculations, instead consider a simpler but reasonable model. Let us put the residual Columbic potential (25) into Eq.(8) and explore the reliability of the computed results in terms of the perturbative method for a single particle state instead of the two-particle-states. A gross estimation for the reliability which we are concerning can be obtained accordingly.

The equation to be solved should have the form[3]:

$$[-\frac{\nabla^2}{E+M_q}-\frac{\alpha_s^{eff}}{r}+A^2r^2]\chi_q^{new}=(E-M_q)\chi_q^{new} \tag{27}$$

where $\frac{1}{2}(E+M_q)$ is the dynamical effective mass of the quark. Eq. (27) can be rewritten as

$$[-\nabla^2-\frac{\lambda}{r}+A^2r^2(E+M_q)]\chi_q^{new}=(E^2-M_q^2)\chi_q^{new} \tag{28}$$

where $\lambda=(E+M_q)\alpha_s^{eff}$.

It is obvious that, if we consider $V^{per}=-\frac{\lambda}{r}$ as a perturbative potential, the 0-th approximation of Eq. (28) is just the same as Eq.(8). For simplicity, let us take the normalized solution χ_q for Eq.(8) given in Ref. [1] as the 0-th order approximate wave function and assume $\Omega_0=\Omega_1=\Omega$. For the $1S$ and $2S$ states, the corresponding 0-th order approximate eigenvalues are

$$em_{1S}^{(0)}=E_0^2-M_q^2=3\Omega, \tag{29}$$

and

$$em_{2S}^{(0)}=E_1^2-M_q^2=7\Omega, \tag{30}$$

respectively.

We can analytically derive the first-order correction to the eigenvalue $\langle\chi_q|V^{per}|\chi_q\rangle$ and the results are

$$em_{1S}^{(1)}=-(E_0+m_q)\alpha_s^{eff}(2\sqrt{\frac{\Omega}{\pi}}) \tag{31}$$

and

$$em_{2S}^{(1)}=-(E_1+m_q)\alpha_s^{eff}(\frac{5}{3}\sqrt{\frac{\Omega}{\pi}}). \tag{32}$$

Besides, we can also calculate the matrix element of the perturbative potential V^{per} in the $\{\chi_q\}$ representation, which reads

$$V_{12}^{per}\equiv\langle\chi_q(1S)|V^{per}|\chi_q(2S)\rangle=-(E+M_q)\alpha_s^{eff}(\sqrt{\frac{2}{3}}\sqrt{\frac{\Omega}{\pi}}). \tag{33}$$

In Ref. [1], the α_s^{eff} is a running coupling constant depending on both flavor and energy state and it is determined by a combination of several complicated relations (Eqs.(13)-(17) of Ref. [1]). For the ground state of a meson, a gross estimation indicates that it is about $0.1 \sim 0.2$. For convenience, we take $\alpha_s^{eff} = 0.15$ instead of the running coupling constant in our calculation below.

Using the data given in [1] as

$$A = 2166\ (MeV)^{\frac{2}{3}}, M_u = M_d = 82.8MeV,$$

$$M_s = 357.5MeV, M_c = 1428MeV, M_b = 4636.6MeV,$$

we can obtain the numerical results of $em_{1S}^{(0)}$, $em_{1S}^{(1)}$, and V_{12}^{per}, we list them below in Table 1.

Table 1

Numerical results of 0-th and 1-st approximate eigenvalues.

flavor	$\Omega(MeV^2)$	$\lambda(MeV)$	$em^{(0)}(MeV^2)$	$em^{(1)}(MeV^2)$	$\frac{\lvert em^{(1)}\rvert}{em^{(0)}}$	$V_{12}^{per}(MeV^2)$	$\frac{\lvert V_{12}^{per}\rvert}{4\Omega}$
u	46780.8	69.9696	140342	-17076.5	0.121678	-6971.44	0.037256
s	65992.9	139.242	197979	-40362.1	0.203871	-16477.8	0.062423
c	118144	446.270	354432	-173085	0.488345	-70661.5	0.149524
b	209335	1401.06	628005	-723324	1.15178	-295296	0.352659

As it is well known [4], if the approximation makes sense, the following conditions must be respected,

$$|em^{(1)}| \ll em^{(0)} \tag{34}$$

and

$$|V_{12}^{per}| \ll em_{2S}^{(0)} - em_{1S}^{(0)} = 4\Omega. \tag{35}$$

It is easy to check from the values in the Table 1 that $\frac{|em_{1S}^{(1)}|}{em_{1S}^{(0)}}$ is about 0.5 for c quark and turns to be larger than 1 for b quark. It means that the first condition (34) is broken seriously for c and b quark. Because $\frac{|V_{12}^{per}|}{em_{2S}^{(0)} - em_{1S}^{(0)}}$ is about 0.35 for b quark , the

second condition (35) does not hold in the $b-$case either. The breakdown of the constraint conditions would undoubtedly undermine the reliability of the numerical results of [1].

To understand this problem is not difficult, because the Coulomb potential is a short range potential. For the heavy quark it is more important than the confinement potential. Our results indicate that it is not appropriate to consider the confinement potential as the 0-th order and the Coulomb potential as a perturbation for a heavy quark system.

4. On the wave function at the origin.

In order to compute the leptonic decay width of vector mesons, the authors of Ref.[1] used the radial wave function of meson evaluated at center $R^{q_iq_j}_{nS}(0)$ (given in eq. (26) of [1]). It means that they only considered the contribution from the relativistic harmonic mean field potential and neglected the residual Coulomb potential. However, in fact the later is the potential which makes the main contribution in the short range and must predominate the value of the wave function at origin. We would like to give more discussions on this aspect by using the simple single-particle example described in Eq. (28) above. The square of the wave function at origin for nS states can be obtained by using the well-known expression[5]

$$|\chi^{new}_q(0)|^2 = \frac{\mu}{2\pi}\langle nS|\frac{dV}{dr}|nS\rangle, \tag{36}$$

where μ is the mass of the particle. For the case corresponding to Eq. (28), we have $2\mu = 1$. If the Coulomb potential is omitted, the result from Eq. (36) is just the same as that computed in Eq.(8) of ref.[1]. Thus, the corresponding squared wave function at origin for 1S state is

$$|\chi^{(0)}_{q,1S}(0)|^2 = \langle\chi^{(0)}_{q,1S}|\frac{d}{dr}(\Omega_0^2r^2)|\chi^{(0)}_{q,1S}\rangle = (\frac{\Omega_0}{\pi})^{\frac{3}{2}}. \tag{37}$$

Including contributions of the Coulomb potential, by using the 0-th wave function of the Eq.(28) we can approximately estimate the contribution of the residual Coulomb potential

to the wavefunction at the origin, which reads

$$|\chi^{(1)}_{q,1S}(0)|^2 = \langle\chi^{(0)}_{q,1S}|\frac{d}{dr}(-\frac{\lambda}{r})|\chi^{(0)}_{q,1S}\rangle = \lambda(\frac{\Omega}{\pi}). \tag{38}$$

Still taking $\alpha^{eff} = 0.15$ and in terms of the values of Table 1 we can obtain the numerical results for $|\chi^{(0)}_{q,1S}(0)|^2$ and $|\chi^{(1)}_{q,1S}(0)|^2$. They are shown in Table 2.

Table 2

Numerical results of 0-th and 1-st approximations for the squared wave function at origin for $1S$ state.

flavor	$\|\chi^{(0)}_{q,1S}(0)\|^2(10^6 MeV^3)$	$\|\chi^{(1)}_{q,1S}(0)\|^2(10^6 MeV^3)$
u	1.81709	1.04190
s	3.04454	2.92494
c	7.29277	16.7825
b	17.2004	93.3574

The data in Table 2 clearly indicate that the contribution of the so-called residual Coulomb potential to the wavefunction is very close to that of the harmonic mean field for light flavors, moreover, it turns much larger for heavy quarks b and c. The situation is much more serious than for eigenvalues. Therefore, in any case, it is not plausible to neglect the Coulomb potential at the 0-th order, even though it is a residual one.

In summary, we think that the idea to construct a unified scheme for flavored mesons and baryons is very desirable and the authors of ref.[1] have made a new instructive trial. However, while solving the basic equation established by the authors and calculating the residual Coulomb energy, as well as the wave function at origin there are some inconsistency and even mistakes, which affect the reasonability and correctness of their numerical results and may destroy the reliability of the solution. Therefore, even though the motivation of Ref.[1] is great, the adopted method for obtaining solution needs to be reconsidered more carefully.

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